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Phase-space dynamical model of an open system of interacting q-bits

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Abstract
The path integral form of the propagator for a stochastic Schrödinger equation is used to derive stochastic differential equations on the quantum phase space of interacting q-bits in a random environment. The equations can be considered as an approximate 'classical' model of the quantum system. Predictions of the model are compared with that of the exact quantum equations for various examples of the environment operators.

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1. Introduction

The development of quantum information technology (QIT) [1] has revitalized interest in the notorious problems of the relation between the quantum and the classical descriptions of a physical system. Robustness and decoherence of quantum superpositions between classically distinct states of the system, entanglement between subsystems and the qualitative properties of the quantum versus the classical dynamics are examples of related phenomena which must be successfully described and understood before the advantages of quantum systems can be used for an efficient processing of information. The standard theory of closed quantum systems does not answer these types of questions. In particular, the effect of the unavoidable coupling between a quantum system and its environment, which is crucial for the control of the quantum processor, could be quite different depending on the qualitative properties of the intrinsic dynamics of the system (see, for example, [2–4]). Analysis of this question is further complicated by the fact that in the case of a genuinely quantum system, like the system of q-bits, which is not obtained by quantization of a classical one, the classical notions of regular or chaotic dynamics have no universally accepted definitions [5, 6]. Typically quantum effects, like dynamical generation of entanglement [7], make the separation of different effects even more difficult. In this paper, we shall show how a phase-space representation of the q-bits, provided by the theory of generalized coherent states [8, 9], can be useful in analysing the influence of noise in qualitatively different dynamical states of the system.

The following Hamiltonian, of N interacting q-bits in an external classical field, could be considered as representing an ideal universal quantum processor [10, 11], which models different experimental proposals:

$$H = \sum_j \mathbf{b}_j \mathbf{S}^j + \sum_{i<j} \lambda_{i,j} \mathbf{S}^i \mathbf{S}^j, \tag{1}$$

where $\mathbf{S}^i\mathbf{S}^j \equiv S_x^i S_x^j + S_y^i S_y^j + S_z^i S_z^j$ and $S_{x,y,z}^j$ are the three components of the vector operator $\mathbf{S}_j$ of the jth q-bit, and satisfy the usual $SU(2)$ commutation relations. The parameter $\mathbf{b}_j$ could be considered as a local magnetic field and $\lambda_{i,j}$ represents the interaction between the i and j spins. These parameters are usually functions of time, bringing an explicit time dependence in $\hat{H}$. For example, the Hamiltonian of the universal quantum processor realized by an array of quantum dots is of the form (1) [11, 12]. Of course, the Hamiltonians of the form (1) have been used to model various systems in solid-state physics for a long time. In particular, and in relation to potential applications in QIP, a variety of interacting charge q-bits based on Josephson junctions are described by Hamiltonian of the form (1), with the corresponding interpretation of the parameters $\mathbf{b}_j$ and $\lambda_{i,j}$. Various theoretical (see, for example, [13–15]) and experimental [16] results have been reviewed recently in [17].

The influence of the environment on the system of q-bits (or its subsystem) in a quantum computer cannot be avoided, and at the same time it is crucial in the relation between the quantum and the classical. The theory of open quantum systems gives the necessary conceptual framework and the required mathematical tools [18–20]. Furthermore, in order to make the quantum and the classical formalisms as close as possible it is convenient to use the phase-space representation of the quantum system. Such representation exists for a large class of quantum systems, even for those, like (1), that are not obtained by the quantization of a classical mechanical model [8, 9, 21]. In this case, the phase-space picture enables one to use precise notions and tools of the theory of dynamical systems to study typically quantum objects [22]. In fact, both the phase-space representations and the theory of open quantum systems have been developed mostly inspired by the common field of quantum optics, and have influenced each other [20].

The states and the dynamics of an open system are most commonly described by considering the reduced density matrix $\hat{\rho}_c$ and the corresponding master equation, obtained by taking the trace over the environmental degrees of freedom [20]. A pair of interacting q-bits in interaction with a bath of quantum oscillators was studied in detail using this approach in [13] (where further references to the solid-state applications can be found). A recent application of a master equation for the reduced density matrix of coupled pair of charged q-bits in Josephson junctions is described in [15]. In the density matrix approach, the primary object is an ensemble of quantum systems. The phase-space representations using various quantum distributions and the corresponding master equations have been used intensively in the past. However, as in the classical case [23], there is also an alternative approach which centres on the quantum systems in a pure state $|\psi(t)\rangle$, whose time evolution is described by a stochastic Schrödinger equation (SSH) with a c-number noise $\eta(t)$ [24, 25], [20]. Classical average of $|\psi_\eta\rangle\langle\psi_\eta|$ over the stochastic process $\eta(t)$ relates the two approaches, i.e. the master equation and the stochastic Schrödinger equation. Our goal is to obtain and use the phase-space representation of a particular type of the stochastic Schrödinger equation corresponding to the system of q-bits in a random environment. As we shall see, the phase-space representation of the stochastic Schrödinger equation gives complex Ito–Langevein stochastic differential equations for the conjugated variables of the quantum phase space. The phase-space equations can be considered as a classical or, better, a phase space model of the quantum system. We shall analyse relations between the exact quantum system and the phase-space model for some examples of the environment operators. The phase-space picture of open systems, described by stochastic Schrödinger equations, have been used before in the cases of linear systems [26, 27] or nonlinear oscillators [2], for which the semi-classical approximation and the classical limit make sense. However, we are not aware of any results along these lines, concerning the important system of q-bits described by (1).

2. q-bit coherent states and the phase space

In this section, we shall briefly describe the quantum phase space of the systems with $SU(2)$ and $SU(2)^{\otimes N}$ dynamical groups and collect the relevant formulae. The general references for the subject, were most that is recollected in this section can be found, are [8, 9].

The dynamical group of a quantum system is a Lie group G such that the state space of the quantum system is also a space of an irreducible unitary representation of G. For example, a free one-dimensional quantum particle and the quantum oscillator have the Heisenberg group as their dynamical group. The dynamical group of a q-bit is $SU(2)$.

The Lie algebra $su(2)$ is given by the commutation relations:

$$[S_z, S_\pm] = \pm S_\pm, \qquad [S_-, S_+] = -2S_z, \tag{2}$$

where $S_z, S_\pm$ are related to x, y, z components of the spin S_x, S_y, S_z by $S_\pm = S_x \pm \mathrm{i}S_y$ and $S_z = S_3$. The systems of units used is such that $\hbar = 1$, since no quasi-classical asymptotic will be involved. The quotient space $M = SU(2)/U(1)$ is obtained by exponentiation of the $su(2)/u(1)$, where $u(1)$ is the maximal commutative sub-algebra of $su(2)$ generated by S_z. M is a symmetric space, with Riemannian and symplectic structure. These structures are explicitly constructed using the generalized coherent states, which are given by the group and (in general) by its particular representation.

For the $SU(2)$ group, the space of an irreducible representation is finite-dimensional space V^s with $\dim V^s = 2s + 1$, where s is fixed non-negative integer or half-integer. In our case $s = 1/2$, but we shall, nevertheless, keep the explicit dependence on s. The formal limit $s \to \infty$ corresponds to a 'classical spin'.

Action of an arbitrary group element $g \in G$ on the lowest weight vector $|s, -s\rangle$ can be split into a product of two terms:

$$\begin{aligned}\hat{T}^s(g)|s,-s\rangle &= \hat{T}^s(d)\hat{T}^s(h)|s,-s\rangle \\ &= \hat{T}^s(d)|s,-s\rangle \exp \mathrm{i}\psi(h), \qquad \exp \mathrm{i}\psi(h) \in U(1)\end{aligned} \tag{3}$$

where $\hat{T}^s$ is a representation matrix.

The quotient manifold $SU(2)/U(1)$ is in a one-to-one correspondence with the set of the coherent states $|\alpha\rangle$, defined by the result of the action of a one parameter family of, so-called, displacement operators $\hat{D}(\alpha)$ on the vector $|s, -s\rangle$. Thus,

$$|\alpha\rangle = \hat{D}(\alpha)|s,-s\rangle = \exp(\alpha\hat{S}_+ - \bar{\alpha}\hat{S}_-)|s,-s\rangle, \tag{4}$$

where α is a complex number, $\bar{\alpha}$ its complex conjugate and $\hat{S}_\pm$ denote the representation matrices in V^s of the algebra generators $S_\pm$.

The manifold $SU(2)/U(1)$ has the structure of a two-dimensional sphere S^2, which can be identified, via the stereographic projection, with $\mathbf{C}\cup\{\infty\}$. The sphere S^2 is a two-dimensional manifold equipped with the standard symplectic structure which is introduced as follows: first, the non-normalized coherent states $\|z\rangle$ are defined by

$$\|z\rangle = \exp z\hat{S}_+|s,-s\rangle, \tag{5}$$

where $z \in \mathbf{C} \cup \{\infty\}$. The norm of $\|z\rangle$ is used to define the so-called Bargman kernel given by

$$K_s(\bar{z}, z) = \ln\langle z\|z\rangle = \ln(1 + |z|^2)^{2s}.$$

The symplectic structure is then given by

$$\omega_s = (-1)^{1/2} \frac{\partial^2 K_s(\bar{z}, z)}{\partial\bar{z}\partial z} \, \mathrm{d}z \wedge \mathrm{d}\bar{z} = \frac{2(-1)^{1/2}}{(1 + z\bar{z})^2} \, \mathrm{d}z \wedge \mathrm{d}\bar{z}, \qquad z \in \mathbf{C} \cup \{\infty\}. \tag{6}$$

The metric on S^2 is given by

$$g_{i,j} = \delta_{i,j} \frac{2s}{(1 + z\bar{z})^2}. \tag{7}$$

The relation between the parameters z and α is obtained via yet another parametrization of the sphere S^2 by angles $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi < 2\pi$. One has

$$z = \tan\frac{\theta}{2} \exp -\mathrm{i}\phi, \qquad \alpha = \frac{\theta}{2} \exp -\mathrm{i}\phi. \tag{8}$$

Local canonical coordinates (q, p) for the symplectic structure ω_s are given by

$$\tau = \left(\frac{1}{4s}\right)^{1/2} (q + \mathrm{i}p) \qquad \bar{\tau} = \left(\frac{1}{4s}\right)^{1/2} (q - \mathrm{i}p), \tag{9}$$

where $\tau = z(1 + z\bar{z})^{-1/2}$. In these coordinates, the Poisson bracket of two functions on S^2 has the canonical form

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}. \tag{10}$$

The sphere S^2, together with the symplectic structure $\omega_{s=1/2}$, is the quantum phase space of a q-bit.

There are various prescriptions, or representations, for constructing a phase-space picture of a quantum system. They are based on the completeness relation for the coherent states

$$\mathbf{1} = \int_{S^2} |\alpha\rangle \, \mathrm{d}\mu(\alpha) \langle\alpha|, \tag{11}$$

where $\mathrm{d}\mu(\alpha) = (2s + 1) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi/4\pi$ is the invariant measure on $S^2 = SU(2)/U(1)$ multiplied by $(2s + 1)$. For our purpose, the Q-representation of the quantum operators is the relevant one. Let $\hat{B}$ be an arbitrary linear operator acting on V^s. Q symbol of the operator $\hat{B}$ is a function on S^2 defined by

$$B_Q(\alpha^*, \alpha) = \langle\alpha|\hat{B}|\alpha\rangle \equiv (\hat{B})_Q \tag{12}$$

where $|\alpha\rangle$ is an $SU(2)$ coherent state, which can be parametrized by the most convenient coordinates. We shall need the Q symbols of the operators $\hat{S}_{\pm,z}$ for the single q-bit system:

$$S_{\pm;Q} = \frac{q \mp \mathrm{i}p}{2}\sqrt{4s - q^2 - p^2} \qquad S_{z,Q} = \frac{1}{2}(q^2 + p^2 - 2s). \tag{13}$$

The linear combinations of operators $\hat{S}_{\pm,z}$ are expressed as the same linear combinations of the corresponding Q symbols. The product of two operators is represented by the convolution of the corresponding symbols. For example, the quadratic operators in the form $\hat{S}^{\dagger}_{\pm,z}\hat{S}_{\pm,z}$ are represented by

$$\begin{aligned} \left(\hat{S}_z^2\right)_Q &= S_{z,Q}^2(p, q) + \frac{1}{8s}(p^2 + q^2)(4s - p^2 + q^2) \\ (\hat{S}_-\hat{S}_+)_Q &= \bar{S}_{-,Q}(p, q) S_{+,Q}(p, q) - \frac{1}{8s}(4s - p^2 q^2)^2, \end{aligned} \tag{14}$$

etc. The Q symbol of the commutators between complex linear combinations of $\hat{S}_{\pm,z}$ is given by the Poisson bracket with the symplectic structure (6) between the corresponding Q symbols. This is not the case for the commutators involving nonlinear expressions of $\hat{S}_{\pm,z}$. Expansion in powers of the parameter $1/s$ of the dynamical equations corresponds to the 'quasi-classical' approximation, but this does not make much sense in the case of a q-bit in a pure state which is a typically quantum object with $s = 1/2$. (However, let us mention that a statistical ensemble of q-bits could be prepared in such a mixture that all expectation values are as in classical statistical mechanics up to a desired accuracy [28].)

A system consisting of several q-bits is described on a space obtained by direct multiplication of the Hilbert spaces corresponding to each of the q-bits, and the phase space is a direct product of the spheres S^2, with the geometric structure of the product: $M = S^2 \times \cdots S^2$. The symplectic and metric structures on M are obtained by repeating the 2×2 blocks (6) and (7) on the diagonal of a $2N \times 2N$ matrix.

The coherent states of N q-bits, $|\alpha\rangle \equiv |\alpha^1\rangle \bigotimes \cdots |\alpha^N\rangle$ are the points of the phase space M, and correspond to disentangled (separable) states of the system. Entangled (non-separable) pure states are described by delocalized distributions on the product of N spheres. The Q symbols of multi q-bit operators are introduced analogously as in the single q-bit case. For example, the following Hamiltonian, of the form (1), contains a sum of bilinear expression $1 \bigotimes \cdots \hat{S}_z^i \cdots \hat{S}_z^j \cdots \bigotimes 1$, and its Q symbol is

$$\langle q, p|\hat{H}|q, p\rangle = \langle q, p| \sum_j b_j S_z{}^j + \sum_{i<j} \alpha_{i,j} S_x^i S_x^j |q, p\rangle$$

$$= \sum_i \frac{b_i}{4}\left(p_i^2 + q_i^2\right) + \sum_{i,j} \frac{\lambda_{i,j}}{4} q_i q_j \sqrt{\left(4s_i - p_i^2 - q_i^2\right)\left(4s_j - p_j^2 - q_j^2\right)}. \tag{15}$$

Stability and bifurcations of the stationary solutions of the Hamiltonian system given by (15) was analysed in [22]. Relation between the symmetry of the quantum Hamiltonian (1) and the qualitative properties of the dynamics of the classical Hamiltonian system given by the Q symbol H_Q was established in [29].

Before we proceed to the discussion of the evolution of the open system in the following sections, let us briefly comment on the phase-space representation of the evolution of the closed system (1). Firstly, consider only one q-bit in the external constant field. The Heisenberg evolution equation for an operator $\hat{B}$ is represented by an evolution equation on the phase space. This is in general a functional integral equation, which can be approximated asymptotically for short time intervals and in an appropriate classical limit by Hamiltonian differential equations. However, if the quantum Hamiltonian is a linear expression of the generators of the dynamical group, as in the case of the Hamiltonian (1) for a single q-bit, the Heisenberg's equations are exactly represented by the Hamiltonian dynamical equations on the phase space. More precisely, if $\hat{B}(t)$ is a solution of the Heisenberg equation with the Hamiltonian $\hat{H}$, which is a linear combination of $\hat{S}_z, \hat{S}_+, \hat{S}_-$, then the function $B_Q(p(t), q(t)) = \langle (p, q)(t)|\hat{B}|(p, q)(t)\rangle = \langle p, q|\hat{B}(t)|p, q\rangle$ is a solution of the Hamilton's classical equation $\dot{B}_Q = \{B_Q, H_Q\}$, where the Hamiltonian $H_Q = \langle p, q|\hat{H}|p, q\rangle$ and the operator $\hat{H}$ are the same linear combinations of the Q symbols $S_{iQ} = \langle p, q|\hat{S}_i|p, q\rangle$ and operators $\hat{S}_i$, respectively. Here, $|p, q\rangle$ are the coherent states, parametrized by the local canonical coordinates, and $|(p, q)(t)\rangle$ is a solution of the corresponding Schrödinger equation. In the linear case, and only in this case, there are no quantum corrections to the solutions of the classical Hamilton equations. This fact can be rephrased by saying that in this (and only this) case the subset of coherent states is invariant under the evolution. In general, an initial coherent state evolves into a pure state which is a nontrivial superposition of coherent states.

In the multi q-bits case, the generators $S^i_{\pm,z} \equiv 1 \bigotimes \cdots \hat{S}^i_{\pm,z} \cdots \bigotimes 1$ for the ith q-bit are represented by functions which depend effectively only on (q_i, p_i). Bilinear products of the generators $\hat{S}^i_{\pm,z}\hat{S}^j_{\pm,z}$, one acting in V_i and the other in V_j, are represented by products of functions $\langle q_i, p_i|\hat{S}^i_{\pm,z}|q_i, p_i\rangle$ and $\langle q_j, p_j|\hat{S}^j_{\pm,z}|q_j, p_j\rangle$. Furthermore, Q-representation of the commutator of any of the components $\hat{S}^{i,j}_{\pm,z}$ with a bilinear expression of the form $\hat{S}^i_{\pm,z} \bigotimes \hat{S}^j_{\pm,z}$ is equal to the Poisson bracket between Q symbol of the operator $\hat{S}^{i,j}_{\pm,z}$ and the Q symbol of $\hat{S}^i_{\pm,z} \bigotimes \hat{S}^j_{\pm,z}$. For example, $\langle \alpha_1, \alpha_2|\left[\hat{S}^1_+, \hat{S}^1_x \bigotimes \hat{S}^2_x\right]|\alpha_1, \alpha_2\rangle$ is equal to the Poisson bracket between Q-representations of the two operators: $\left\{S^1_{+,Q}, S^1_{x,Q}S^2_{x,Q}\right\}$. This is a simple consequence of the bilinear character of the expression $\hat{S}^1_{\pm,z} \bigotimes \hat{S}^2_{\pm,z}$, and the fact that for each single q-bit the Lie algebra relations between the generators and their Q-representations are the same: $\langle \alpha_i|\left[\hat{S}^i_{\pm,z}, \hat{S}^i_{\pm,z}\right]|\alpha_i\rangle = \left\{S^i_{z,\pm,Q}, S^i_{z,\pm,Q}\right\}$. Also, let us point out that the interaction operators of the bilinear form are actually quite general, because any well-behaved function, say polynomial or analytic function of $\mathbf{S}^i\mathbf{S}^j$, can be expressed as $f(\mathbf{S}^i\mathbf{S}^j) = (3f(1) + f(-3))\mathbf{1}/4 + (f(1) - f(-3))\mathbf{S}^i\mathbf{S}^j/4$ [11]. To conclude, the evolution of the Q symbols of the Heisenberg operators for linear combinations of components $\hat{S}^i_{\pm,0}$ could be calculated from some Hamilton equations with the Hamilton's function $H_Q(q_1 \cdots p_N)$. These equations provide an approximate phase-space model of the quantum system, which gives exact results for $S_{i;Q}(t)$. In particular, this enables one to introduce the notions of qualitatively different types of quantum dynamics in parallel with those of the Hamiltonian dynamical system given by $H_Q(q_1 \cdots p_N)$ on the product phase space M [22].

3. Linear stochastic Schrödinger equation for the system of q-bits

Evolution of an individual system of q-bits from a pure initial state $|\psi\rangle \in \mathcal{H} \equiv V^{\bigotimes N}$ and under stochastic influence of the environment can be described by a modification of the standard Schrödinger equation for the closed system with the Hamiltonian (1). There exist different forms of the stochastic Schrödinger equations with c-number or noncommutative noise (instead of a long list of quite important references we cite just the review given in [20]). Different linear stochastic Schrödinger equations (LSSE) have been used in different contexts (see, for example, [25, 30, 31]). In what follows we shall use the following general form of an LSSE with a c-number noise:

$$\mathrm{d}|\psi\rangle = \mathrm{i}\hat{H}|\psi\rangle\,\mathrm{d}t - \frac{1}{2}\sum_l E^\dagger_l E_l|\psi\rangle\,\mathrm{d}t + \sum_l E_l|\psi\rangle\,\mathrm{d}W_l. \tag{16}$$

Equation (16) represents a symbolic form of an Ito stochastic differential equation on the Hilbert space $\mathcal{H}$. The first term represents the standard Schrödinger equation and the last two terms, containing the so-called environment operators $\hat{E}_l$ which act also in $\mathcal{H}$, model the influence of the environment. It is natural to suppose that each $\hat{E}_l$ can be expressed in terms of the spin operators $\hat{S}^i_{\pm,0}$ of a single q-bit. Thus, the environment operators acting on the ith q-bit do not affect directly the dynamics of the $j \neq i$ q-bit.

The first of the two terms due to environment describes the drift of $|\psi(t)\rangle$, and could be included in an effective non-Hermitian Hamiltonian operator

$$\hat{H}_{\mathrm{eff}} = \hat{H} + \frac{\mathrm{i}}{2}\sum_l \hat{E}^\dagger_l\hat{E}_l. \tag{17}$$

The last term describes diffusion of the state vector and contains independent increments (indexed by l) $\mathrm{d}W_l$ of complex Wigner c-number processes $\eta_l(t)$. Thus, we can write, symbolically $\mathrm{d}W_l(t) = \eta_l(t)\,\mathrm{d}t$ and the increments satisfy

$$\mathcal{M}[\mathrm{d}W_l] = \mathcal{M}[\mathrm{d}W_l\,\mathrm{d}W_{l'}] = 0 \qquad \mathcal{M}[\mathrm{d}W_l\,\mathrm{d}\bar{W}_{l'}] = \delta_{l,l'}\,\mathrm{d}t, \tag{18}$$

where $\mathcal{M}$ denotes the mean value over the complex white noise $\eta_i(t)$. The latter satisfy

$$\mathcal{M}[\eta_l] = \mathcal{M}[\eta_l \eta_{l'}] = 0 \qquad \mathcal{M}[\eta_l(t)\bar{\eta}_{l'}(t')] = \delta_{l,l'}\delta(t - t'). \tag{19}$$

The assumption of white noise could be lifted, and the LSSE and nonlinear SSE for the non-Markovian environments have been constructed (see, for example, [32–34].)

The LSSE (16) represents a particular stochastic unravelling of the Lindblad master equation [35, 20], for the reduced density matrix $\hat{\rho}_c(t)$

$$\frac{\mathrm{d}\hat{\rho}_c(t)}{\mathrm{d}t} = -\mathrm{i}[\hat{H}, \hat{\rho}_c] - \frac{1}{2}\sum_l \left[\hat{E}_l\hat{\rho}_c, \hat{E}_l^\dagger\right] + \left[\hat{E}_l, \hat{\rho}_c\hat{E}_l^\dagger\right], \tag{20}$$

in the sense that the solutions of (16) and (20) satisfy

$$\hat{\rho}_c(t) = \mathcal{M}[|\psi(t)\rangle\langle\psi(t)|]. \tag{21}$$

In fact, it can be shown that the form (16) of the LSSE with complex noise $\eta_l(t)$ is the unique linear unravelling of (20) which has the same invariance as (20) under the unitary transformations in the space of the environment operators $\{\hat{E}_l\}$ [24].

The LSSE equation (16) is linear, which is of fundamental importance for our purposes, but it does not preserve the norm of $|\psi\rangle$, i.e.

$$\mathrm{d}\langle\psi|\psi\rangle = 2\sum \mathrm{Re}(\langle\psi|\hat{E}_l|\psi\rangle\,\mathrm{d}W_l). \tag{22}$$

Thus, to get the correct $\hat{\rho}_c(t)$ which satisfies $\mathrm{d}\,\mathrm{Tr}\,\hat{\rho}_c(t) = 0$ one has to use the non-normalized projectors $|\psi(t)\rangle\langle\psi(t)|$ in (21).

The Ito–Schrödinger evolution equation (16) can be used to define an Ito–Heisenberg evolution for operators by requiring $\langle\psi(t)|\hat{S}|\psi(t)\rangle =: \langle\psi|\hat{S}(t)|\psi\rangle$. The equation for a Hermitian $\hat{S}(t)$ is then given by

$$\mathrm{d}\hat{S} = -\mathrm{i}[\hat{H}_{\mathrm{eff}}, \hat{S}]\,\mathrm{d}t + \sum_l \hat{E}_l^\dagger\hat{S}\hat{E}_l\,\mathrm{d}W_l\,\mathrm{d}\bar{W}_l + \hat{S}\sum_l \hat{E}_l\,\mathrm{d}W_l + \sum_l \hat{E}_l^\dagger\,\mathrm{d}\bar{W}_l\hat{S} \tag{23}$$

where Ito rules (18) have been used. The Q symbol of the stochastic increment $\mathrm{d}\hat{S}$ is

$$(\mathrm{d}\hat{S})_Q = -\mathrm{i}([\hat{H}_{\mathrm{eff}}, \hat{S}])_Q\,\mathrm{d}t + \sum_l \left(\hat{E}_l^\dagger\hat{S}\hat{E}_l\right)_Q\mathrm{d}W_l\,\mathrm{d}\bar{W}_l + \sum_l(\hat{S}\hat{E}_l)_Q\,\mathrm{d}W_l + \sum_l \left(\hat{E}_l^\dagger\hat{S}\right)_Q\mathrm{d}\bar{W}_l. \tag{24}$$

Averaging $(\mathrm{d}\hat{S})_Q$ using (18) gives

$$\mathcal{M}\left[\left(\frac{\mathrm{d}\hat{S}}{\mathrm{d}t}\right)_Q\right] = \mathcal{M}\left[-\mathrm{i}([\hat{H}_{\mathrm{eff}}, \hat{S}])_Q + \sum_l \left(\hat{E}_l^\dagger\hat{S}\hat{E}_l\right)_Q\right]. \tag{25}$$

In the following sections we shall obtain a dynamical model on the phase space of the noise multi q-bit system, i.e. a set of stochastic differential equations for the increments of the canonical coordinates of the phase space. These equations can be used to calculate stochastic increments of Q symbols $\mathrm{d}S_Q$ and their averages. Comparing these equations for the Q symbols of the components $S^i_{x,y,z;Q}$ with the quantum equations (24), (25), for various choices of the environment and the system operators, provides an information about the domain of the phase-space model.

Let us mention that there is a nonlinear unravelling of (20) that corresponds to LSSE (16) [24]. There are also other nonlinear stochastic Schrödinger equations that have been useful, in particular, for numerical computations [20]. However, for the path-integral representation of the stochastic evolution of $|\psi(t)\rangle$, the linear character of (16) is fundamental.

4. Path-integral solution of LSSE for the system of q-bits

In this section, we apply the usual procedure for the construction of the path integral to the coherent-state representation of the propagator, i.e. $G(\alpha_1, t; \alpha_0, t_0) \equiv \langle\alpha_1|G(t, t_0)|\alpha_0\rangle$ where $|\alpha_{1,2}\rangle$ are arbitrary multi q-bit coherent states and $\hat{G}(t, t_0)$ is the propagator for the LSSE (16). In order to avoid confusion, we should perhaps mention that the path integral (32) introduced here is different from the Fayman–Vernon path integral commonly used to derive the master equations in the treatment of open systems [36, 37]. We recommend [13] for an application of Fayman–Vernon approach to a pair of q-bits in a bath of oscillators. On the other hand, the problem which we analyse and our results for the path integral are quite similar to those studied in [38], the major difference being that we analyse the system of q-bits in the phase-space representation by coherent states, whereas in [38] the flat R^{2d} phase space is assumed and the ordinary oscillators Wigner functions are used to bring in the phase-space coordinates.

The propagator can be written as a time-ordered product of propagators over small intervals of time. Thus,

$$\begin{aligned} G(\alpha_1, t; \alpha_0, t_0) &= \langle\alpha_1|\prod_{k=1}^{k=n}\hat{T}\exp\left\{\int_{(k-1)\epsilon}^{k\epsilon}\hat{L}(t')\,\mathrm{d}t'\right\}|\alpha_0\rangle \\ &\approx \langle\alpha_1|\prod_{k=1}^{k=n}\left(1 - \mathrm{i}\hat{H}_k\epsilon - \frac{1}{2}\sum_l \hat{E}_l^\dagger\hat{E}_l\epsilon + \hat{E}_l\,\triangle W(k)_l\right)|\alpha_0\rangle, \end{aligned} \tag{26}$$

where

$$\epsilon = \frac{t - t_0}{n}, \qquad \triangle W(k)_l = \eta_l(k\epsilon) - \eta_l((k-1)\epsilon). \tag{27}$$

Because the intervals ϵ are small $\hat{H}(t)$, $t \in [(k-1)\epsilon, k\epsilon]$ is replaced by a constant operator $\hat{H}_k \equiv \hat{H}(t_k)$, $t_k \in [(k-1)\epsilon, k\epsilon]$, and $\hat{E}$ are assumed to have no explicit dependence on time, so that the time ordering operation in the first line becomes redundant.

As usual [39], we introduce in (26) $n-1$ resolutions of unity in terms of the multi q-bit coherent states $\mathbf{1} = \int_{\mathcal{M}}|\alpha\rangle\,\mathrm{d}\mu(\alpha)\langle\alpha|$ ($\mathrm{d}\mu$ is a product of N single q-bit measures but we use the same symbol). The coherent-state propagator becomes

$$\begin{aligned} G(\alpha_1, t, \alpha_0, t_0) = \lim_{n\to\infty}\int\prod_{k=1}^{k=n}\mathrm{d}\mu(\alpha_k)\Bigg[&\langle\alpha_k|\alpha_{k-1}\rangle - \mathrm{i}\langle\alpha_k|\hat{H}_k|\alpha_{k-1}\rangle\epsilon \\ &+ \sum_l \frac{1}{2}\langle\alpha_k|\hat{E}_l^\dagger\hat{E}_l|\alpha_{k-1}\rangle\epsilon - \langle\alpha_k|\hat{E}_l|\alpha_{k-1}\rangle\triangle W(k)\Bigg]. \end{aligned} \tag{28}$$

The scalar product between the coherent states at two consecutive t_{k-1} and t_k can be written as

$$\begin{aligned} \langle\alpha_k|\alpha_{k-1}\rangle &= \langle\alpha_k|\alpha_k\rangle\left(1 - \frac{\langle\alpha_k|\triangle\alpha_k\rangle}{\langle\alpha_k|\alpha_k\rangle}\right) = 1 - \langle\alpha_k|\triangle\alpha_k\rangle \\ &\approx \exp(-\langle\alpha_k|\triangle\alpha_k\rangle), \end{aligned} \tag{29}$$

where we used the fact that the coherent states $|\alpha\rangle$ are normalized.

Substituting (29) into (28), $G(\alpha_1, t, \alpha_0, t_0)$ becomes

$$\begin{aligned} \lim_{n\to\infty}\int\prod_{k=1}^{k=n}\mathrm{d}\mu(\alpha_k)\exp\mathrm{i}\epsilon\Bigg[&\mathrm{i}\left\langle\alpha_k\left|\frac{\mathrm{d}}{\mathrm{d}t}\alpha_k\right.\right\rangle - \langle\alpha_k|H_k|\alpha_k\rangle - \sum_l\frac{\mathrm{i}}{2}\langle\alpha_k|\hat{E}_l^\dagger\hat{E}_l|\alpha_k\rangle \\ &+\mathrm{i}\frac{\triangle W(k)}{\epsilon}\langle\alpha_k|E|\alpha_k\rangle\Bigg]. \end{aligned} \tag{30}$$

The expression $\mathrm{i}\langle\alpha_k\,|\,\frac{\mathrm{d}}{\mathrm{d}t}\alpha_k\rangle$ can be written in terms of the complex coordinates z_i and $\bar{z}_i$ or in terms of the canonical coordinates (q, p) of the coherent states as follows:

$$\mathrm{i}\left\langle\alpha\left|\frac{\mathrm{d}}{\mathrm{d}t}\alpha\right.\right\rangle = \frac{\mathrm{i}}{2}\sum_{i,j}\left(\bar{z}_i\frac{\mathrm{d}z_j}{\mathrm{d}t} - z_i\frac{\mathrm{d}\bar{z}_j}{\mathrm{d}t}\right)g_{i,j} = \sum_i\left(-p_i\frac{\mathrm{d}q_i}{\mathrm{d}t} + q_i\frac{\mathrm{d}q_i}{\mathrm{d}t}\right), \tag{31}$$

where $g_{i,j}$ is the metric on the multi q-bit phase space. The mean values $\langle\alpha|\cdots|\alpha\rangle$ in (30) are the Q symbols of the corresponding operators, which we shall write as functions of $\bar{z}$ and z or of the canonical coordinates (q, p).

With these substitutions and in the limit $n \to \infty$ the expression (30) can be formally written in the form of a path integral

$$G(\bar{z}_1, t, z_0, t_0) = \int \mathcal{D}(z(t))\exp\mathrm{i}S(\bar{z}(t), z(t)), \tag{32}$$

where $\mathcal{D}$ is a measure over the space of paths on the phase space (with the usual reservations concerning the existence of the limit). The expression $S(\bar{z}, z)$ is given by

$$S = \int_{t_0}^{t_1}\mathrm{d}t\left(\frac{\mathrm{i}}{2}\sum_{i,j}g_{i,j}\left(\bar{z}_i\frac{\partial z_j}{\partial t} - z_i\frac{\partial\bar{z}_j}{\partial t}\right) - H_Q(\bar{z}, z) - \frac{\mathrm{i}}{2}\sum_l\left(\hat{E}_l^\dagger\hat{E}_l\right)_Q(\bar{z}, z)\right)$$
$$+\mathrm{i}\int_{t_0}^{t_1}\sum_l\mathrm{d}W_l(\hat{E}_l)_Q(\bar{z}, z) \tag{33}$$

in terms of the Riemann integral $\int_{t_0}^{t_1}\mathrm{d}t$ and the stochastic integral $\int_{t_0}^{t_1}\mathrm{d}W$ represents the action for the LSSE (16). It is a complex and stochastic functional over the phase-space paths.

An interpretation of the functional in (32) can be obtained by writing explicitly the real part

$$\mathrm{Re}\,S = \int_{t_0}^{t_1}\mathrm{d}t\left[\sum_i\left(-p_i\frac{\mathrm{d}q_j}{\mathrm{d}t} + q_i\frac{\mathrm{d}p_i}{\mathrm{d}t}\right) - H_Q(p, q)\right] - \mathrm{Im}\left(\sum_l\int_{t_0}^{t_1}(\hat{E}_l)_Q\,\mathrm{d}W_l\right) \tag{34}$$

and the imaginary part

$$\mathrm{Im}\,S = \int_{t_0}^{t_1}\mathrm{d}t\frac{-1}{2}\sum_l\left(\hat{E}_l^\dagger\hat{E}_l\right)_Q + \mathrm{Re}\left(\sum_l\int_{t_0}^{t_1}(\hat{E}_l)_Q\,\mathrm{d}W_l\right) \tag{35}$$

of the action. We see that the integrand in (32) is a product of two terms; the usual exponent of a phase $\exp\mathrm{i}\,\mathrm{Re}\,S$ and the weight factor $\exp-\mathrm{Im}\,S$. The extremal paths of (33) satisfy the stochastic differential equations on the phase space which are of the form of Hamilton's equations, with the Hamiltonian H_Q, and the noise term $\mathrm{Im}\big(\sum_l(\hat{E}_l)_Q\eta_l\big)$. The weight functional $\exp-\mathrm{Im}\,S$ associates different weights to contributions of different paths to the path integral. The most favoured paths satisfy a relation between the stochastic fluctuations $\eta(t)$ and the values of the Q symbols of the environment operators.

5. Stochastic differential equations on the phase space

The method of stationary exponent approximation for the integral (32) consists in replacing the action functional $S(\bar{z}(t), z(t))$ by its expansion in small deformations from the extremal paths. The zero-order approximation gives Gaussian integrals that can be solved, and better approximation of the integral is obtained by including the further terms in the expansion if the integrals can be calculated. However, we shall not be interested in performing an approximate integration of the path integral (32), but in the phase-space equations which determine the extremal paths. The latter are determined by the following complex Langevin–Ito equations:

$$\begin{aligned} \mathrm{i}g_{i,j}\,\mathrm{d}z_j &= \frac{\partial}{\partial \bar{z}_j}\left(H_Q(\bar{z},z) - \frac{\mathrm{i}}{2}(\hat{E}^\dagger\hat{E})_Q(\bar{z},z)\right)\mathrm{d}t + \mathrm{i}\frac{\partial(\hat{E})_Q(\bar{z},z)}{\partial \bar{z}_j}\,\mathrm{d}W, \\ \mathrm{i}g_{i,j}\,\mathrm{d}\bar{z}_j &= -\frac{\partial}{\partial z_j}\left(H_Q(\bar{z},z) - \frac{\mathrm{i}}{2}(\hat{E}^\dagger\hat{E})_Q(\bar{z},z)\right)\mathrm{d}t - \mathrm{i}\frac{\partial(\hat{E})_Q(\bar{z},z)}{\partial z_j}\,\mathrm{d}W, \end{aligned} \tag{36}$$

where we have included the sum over different channels $\sum_l$ in the symbol $\hat{E}$. The expression $(\hat{E}^\dagger\hat{E})_Q(\bar{z},z))$ is the Q symbol of the sum of the products of operators $\hat{E}_l^\dagger\hat{E}_l$. It differs from the product of the Q symbols $(\hat{E}^\dagger)_Q(\hat{E})_Q$ by a function proportional to $1/s$, like in the example (14).

In terms of the $2N$ canonical coordinates (q,p), the equations are

$$\begin{aligned} \mathrm{d}q_j &= \frac{\partial}{\partial p_j}\left(H_Q(q,p) - \frac{\mathrm{i}}{2}(\hat{E}^\dagger\hat{E})_Q(q,p)\right)\mathrm{d}t + \mathrm{i}\frac{\partial(\hat{E})_Q(q,p)}{\partial p_j}\,\mathrm{d}W \\ \mathrm{d}p_j &= -\frac{\partial}{\partial q_j}\left(H_Q(q,p) - \frac{\mathrm{i}}{2}(\hat{E}^\dagger\hat{E})_Q(q,p)\right)\mathrm{d}t - \mathrm{i}\frac{\partial(\hat{E})_Q(q,p)}{\partial q_j}\,\mathrm{d}W. \end{aligned} \tag{37}$$

The symbol $\mathrm{d}W$ in general denotes a four-dimensional process whose properties depend on the particular form of the noise $\sum_l \hat{E}_l\,\mathrm{d}W_l$, as will be illustrated in the next section. We shall use the general Ito formula for the stochastic increment of a function S_Q

$$\begin{aligned} \mathrm{d}S_Q = \sum_{i=1}^{4}\left[A_i\frac{\partial S_Q}{\partial x_i} + \bar{A}_i\frac{\partial S_Q}{\partial x_i}\right]\mathrm{d}t + \sum_{i,j} B_{i,j}\frac{\partial S_Q}{\partial x_i}\,\mathrm{d}W_i j \\ + \sum_{i,j}\bar{B}_{i,j}\frac{\partial S_Q}{\partial x_i}\,\mathrm{d}\bar{W}_j + \frac{1}{2}\sum_{i,j}^{4}(BB^\dagger)_{i,j}\frac{\partial^2 S_Q}{\partial x_i\partial x_j}\,\mathrm{d}t, \end{aligned} \tag{38}$$

where the four-dimensional stochastic process

$$\mathrm{d}x_i = A_i(x_1,x_2,x_3,x_4)\,\mathrm{d}t + \sum_j B_{i,j}(x_1,x_2,x_3,x_4)\,\mathrm{d}W_j, \tag{39}$$

is defined by (37) via identification: $x_1 \equiv q_1, x_2 \equiv q_2, x_3 \equiv p_1, x_4 \equiv p_2$.

Averaging $\mathrm{d}S_q$ results in

$$\mathcal{M}\left[\frac{\mathrm{d}S_Q}{\mathrm{d}t}\right] = \sum_{i=1}^{4}\mathcal{M}\left[A_i\frac{\partial S_Q}{\partial x_i} + \bar{A}_i\frac{\partial S_Q}{\partial x_i} + \frac{1}{2}\sum_j (BB)_{i,j}^\dagger\frac{\partial^2 S_Q}{\partial x_i\partial x_j}\right]. \tag{40}$$

Equations (37) could be interpreted as a classical approximate model of the multi q-bit quantum system. The Hamilton function H_Q and the functions corresponding to the drift $(\hat{E}^\dagger\hat{E})_Q$ and the diffusion $(\hat{E})_Q$ are exact coherent-state representations of the quantum operators. However, because neither the Hamilton's function nor the drift term are quadratic functions of the canonical coordinates, the relation between the stationary paths, i.e., the model (37) and the quantum system is not clear and should be investigated. For example, as was pointed out before, in the case of the isolated system of q-bits (1) the evolution of Q symbols of the Heisenberg operators $\left(\hat{S}^i_{x,y,z}(t)\right)_Q$ can be calculated using the phase-space equations, i.e. $\left(\mathrm{d}\hat{S}^i_{x,y,z}(t)\right)_Q = \mathrm{d}S^i_{x,y,z;Q}$, which are real and Hamiltonian in this case [22]. In what follows we shall analyse the relation between the truly quantum evolution of the Q symbols of the components $\left(\hat{S}^i_{x,y,z}\right)_Q$ and the evolution of these functions according to equations (37), for some special forms of the environment operators.

6. Examples

In the following examples, we shall consider the system of two q-bits, which is sufficient to have a realization of a universal quantum processor [40, 1]. In fact, we shall consider the pair of q-bits with the Hamiltonian

$$\hat{H} = \lambda \hat{S}^1_x \hat{S}^2_x + b\hat{S}^1_z + b\hat{S}^2_z, \tag{41}$$

where b and λ are constants. The evolution of Q symbols of any of the components $\hat{S}^i_{x,y,z}$ for the isolated system (41) is exactly reproduced by the corresponding phase-space equation and shows typical properties of a Hamiltonian dynamical system with mixed dynamics [22].

In order to compare the truly quantum and the phase-space evolutions, we compare a formula for the stochastic increment $\mathrm{d}S_Q$ of a Q symbol, generated by the Ito–Langevien equations (38) and (40) with the quantum formulae (24) and (25) for $(\mathrm{d}\hat{S})_Q$. Let us also mention that in order to preserve the original interpretation of the model the limit $s \to \infty$ should not be used. The value of each of the spins is held fixed $s^i = 1/2$.

We shall suppose that the stochastic term in LSSE (16) is of the following form:

$$\sum_{i=1}^{2} \mathrm{d}W^i_x \hat{S}^i_x + \mathrm{d}W^i_y \hat{S}^i_y + \mathrm{d}W^i_y \hat{S}^i_y, \tag{42}$$

where the six processes $\mathrm{d}W^i_{x,y,z}$ are assumed to be independent. In this case, the Ito rules for the components $\mathrm{d}W_{q_i}$ and $\mathrm{d}W_{p_i}$ in equations (37) are

$$M\big[\mathrm{d}\bar{W}_{q_i}\,\mathrm{d}W_{q_j}\big] = \delta_{i,j}\,\mathrm{d}t, \qquad M\big[\mathrm{d}\bar{W}_{p_i}\,\mathrm{d}W_{q_j}\big] = \delta_{i,j}\,\mathrm{d}t, \qquad M\big[\mathrm{d}\bar{W}_{p_i}\,\mathrm{d}W_{p_j}\big] = \delta_{i,j}\,\mathrm{d}t. \tag{43}$$

The Ito formula (38) in our case, because of (37) and (43) and with identification $x_1 \equiv q_1, x_2 \equiv q_2, x_3 \equiv p_1, x_4 \equiv p_2$, becomes

$$\mathrm{d}S_Q = \sum_{i=1}^{4}\left[A_i \frac{\partial S_Q}{\partial x_i} + \bar{A}_i \frac{\partial S_Q}{\partial x_i}\right]\mathrm{d}t + \sum_i B_i \frac{\partial S_Q}{\partial x_i}\,\mathrm{d}W_i + \sum_i \bar{B}_i \frac{\partial S_Q}{\partial x_i}\,\mathrm{d}\bar{W}_i + \frac{1}{2}\sum_i^4 (BB^\dagger)_{i,i} \frac{\partial^2 S_Q}{\partial x_i^2}\,\mathrm{d}t. \tag{44}$$

Let us now apply the formulae (37) for the Hamiltonian and the noise terms of the forms (41) and (42). The Q symbols of the Hamiltonian and the quadratic operators entering the drift term are of the following form:

$$\begin{aligned}
H_Q &= \frac{1}{2}\sum_{i=1,2}\left(p_i^2 + q_i^2\right) + \mu q_1 q_2 \sqrt{\left(2 - p_1^2 - q_1^2\right)\left(2 - p_2^2 - q_2^2\right)},\\
\left(\hat{S}^\dagger_z \hat{S}_z\right)_Q &= \frac{1}{4}[(q^2 + p^2 - 1)^2 + (p^2 + q^2)(2 - p^2 - q^2)] = 1/2\\
\left(\hat{S}^\dagger_x \hat{S}_x\right)_Q &= \frac{1}{4}[4 - 2p^4 - 2q^2 + q^4 + p^2(4 - 3q^2)]\\
\left(\hat{S}^\dagger_z \hat{S}_z\right)_Q &= \frac{1}{4}[-3p^4 + p^2(6 + q^2) + 2(-2 - 2q^2 + q^4)],
\end{aligned} \tag{45}$$

where we have set $s = 1/2$ and $\mu = \lambda/2b$.

Equations (37) for the complex stochastic increments of the canonical coordinates become

$$\begin{aligned} \mathrm{d}q_i &= Q_i(q_1, q_2, p_1, p_2)\,\mathrm{d}t + DQ_i(q_i, p_i)\,\mathrm{d}t + \mathrm{Diff}\ Q_i(q_i, p_i)\,\mathrm{d}W_{q_i} \\ \mathrm{d}p_i &= P_i(q_1, q_2, p_1, p_2)\,\mathrm{d}t + DP_i(q_i, p_i)\,\mathrm{d}t + \mathrm{Diff}\ P_i(q_i, p_i)\,\mathrm{d}W_{p_i}, \end{aligned} \tag{46}$$

where the functions Q_i, P_i, DQ_i, DP_i, Diff Q_i and Diff P_i are given by

$$Q_1 = p_1 - \mu p_1 q_1 q_2 \frac{\sqrt{2 - p_2^2 - q_2^2}}{\sqrt{2 - p_1^2 - q_1^2}},$$

$$P_1 = -q_1 + \mu q_1 q_1 q_2 \frac{\sqrt{2 - p_2^2 - q_2^2}}{\sqrt{2 - p_1^2 - q_1^2}} - 2\mu q_2 \sqrt{\left(2 - p_2^2 - q_2^2\right)\left(2 - p_1^2 - q_1^2\right)},$$

$$DQ_1 = \frac{\mathrm{i}}{8}\left(-20p_1^3 - 4p_1\left(-5 + q_1^2\right)\right)$$

$$DP_1 = \frac{\mathrm{i}}{8}\left(12q_1^3 - 4p_1^2 q_1 - 12q_1\right),$$

$$\mathrm{Diff}\ Q_1 = \frac{\mathrm{i}}{2}\left[\frac{-p_1 q_1\,\mathrm{d}W_{x_1}}{\sqrt{2 - p_1^2 - q_1^2}} - \frac{p_1^2\,\mathrm{d}W_{y_1}}{\sqrt{2 - p_1^2 - q_1^2}} - \sqrt{2 - p_1^2 - q_1^2}\,\mathrm{d}W_{y_1} + 2p_1\,\mathrm{d}W_{z_1}\right],$$

$$\mathrm{Diff}\ P_1 = \frac{\mathrm{i}}{2}\left[\frac{q_1^2\,\mathrm{d}W_{x_1}}{\sqrt{2 - p_1^2 - q_1^2}} - \sqrt{2 - p_1^2 - q_1^2}\,\mathrm{d}W_{x_1} + \frac{p_1 q_1\,\mathrm{d}W_{y_1}}{\sqrt{2 - p_1^2 - q_1^2}} - 2p_1\,\mathrm{d}W_{z_1}\right],$$

and analogously for the second q-bit.

The first observation that we would like to make concerns the situation when the environment is coupled only to one of the two q-bits, say the first q-bit. This situation is used in a modelling of the decoherence on a QI processor in a recent paper [41]. The evolution of the Q symbols $S^2_{x,y,z;Q}$ of the second q-bit is then exactly described by the model (37) and equations (40), so that the phase-space model is in this sense correct, and could be used to complement the analyses in [41]. In fact, the exact quantum equations for $\mathcal{M}[(\mathrm{d}\hat{S}^2)_Q/\mathrm{d}t]$ (25) and the model equation for $\mathcal{M}\left[\mathrm{d}\hat{S}^2_Q\big/\mathrm{d}t\right]$ (40) have only the Hamiltonian terms and these are equal. The stochastic influence of the environment is felt through the coupling with the first q-bit.

The other case, in which the relations between the exact quantum evolution and the phase-space model can easily be understood, analytically occurs when the environment operators are expressed as linear combinations of the z components $\hat{S}_z^{1,2}$. Let us point out that z component of the noise does not contribute to the drift term. In this case, i.e. if the noise is $\sum_{1,2} \hat{S}_z^i\,\mathrm{d}W_{z_i}$, then the equations simplify to

$$\mathrm{d}q_i = Q_i\,\mathrm{d}t + ip_i\,\mathrm{d}W_{z_i}, \qquad \mathrm{d}p_i = P_i\,\mathrm{d}t - iq_i\,\mathrm{d}W_{z_i}. \tag{47}$$

In this case, the Ito formulae (44) for the components $\mathrm{d}S_{x;Q}$; $\mathrm{d}S_{y;Q}$ and $\mathrm{d}S_{z;Q}$ are

$$\begin{aligned} \mathrm{d}S^1_{z,Q} = \mu p_1 q_2 \sqrt{\left(2 - p_2^2 - q_2^2\right)\left(2 - p_1^2 - q_1^2\right)} + \mathrm{i}\,\overbrace{(p_1 q_1 - p_1 q_1)}^{=0}\left(\mathrm{d}W_{z_1} + \mathrm{d}\bar{W}_{z_1}\right) \\ + \frac{\left(p_1^2 + q_1^2\right)}{2}\,\mathrm{d}W_{z_1}^2, \end{aligned} \tag{48}$$

$$\mathrm{d}S^1_{x,Q} = \frac{1}{2}p_1\sqrt{2-p_1^2-q_1^2}\left(1+\mathrm{i}\,\mathrm{d}W_{z_1}-\mathrm{i}\,\mathrm{d}\bar{W}_{z_1}\right) + \frac{q_1\left(3p_1^4+2p_1^2\left(-3+q_1^2\right)+q_1^2\left(-2+q_1^2\right)\right)}{4\left(2-p_1^2-q_1^2\right)^{3/2}}\,\mathrm{d}W_{z_1}^2 \tag{49}$$

$$\mathrm{d}S^1_{y,Q} = \frac{1}{2}q_1\sqrt{2-p_1^2-q_1^2}\left(1+\mathrm{i}\,\mathrm{d}W_{z_1}-\mathrm{i}\,\mathrm{d}\bar{W}_{z_1}\right) - \mu\left(p_1^2q_2-q_2+q_1^2q_2\right)\sqrt{2-p_2^2-q_2^2} - \frac{p_1\left(p_1^4+3q_1^2\left(-2+q_1^2\right)+2p_1^2\left(-1+q_1^2\right)\right)}{4\left(2-p_1^2-q_1^2\right)^{3/2}}\,\mathrm{d}W_{z_1}^2. \tag{50}$$

Comparison of equations (48)–(50) with the corresponding exact quantum equations (24) leads to the following conclusions. Firstly, if the environment operator is $\hat{E}_1 = \hat{S}^1_z$, $\hat{E}_2 = \hat{S}^2_z$, then the exact quantum equation (24) and the model equations (48) give the same evolution of $S^{1,2}_{z,Q}$, In fact, in this case the drift term is zero and there are only the Hamiltonian and the diffusion terms.

Consider now the evolution of a component $S^1_{x,Q}$ with the environment like in equations (47). The Hamiltonian terms, and the terms with $\mathrm{d}W_{z_1}$ and $\mathrm{d}\bar{W}_{z_1}$ in the Ito equation (49), coincide with the corresponding terms in the exact quantum equation. However, there is no drift term in the model equations, and the exact quantum equations have a nonzero drift. In fact, the Q symbol of $(\hat{S}_z\hat{S}_z)$ is constant and consequently $\{(\hat{S}_z\hat{S}_z)_Q, S_{x,Q}\} = 0$, which is different from the Q symbol of the commutator $[\hat{S}_z\hat{S}_z, \hat{S}_x] = \mathrm{i}((\hat{S}_z\hat{S}_y)_Q + (\hat{S}_y\hat{S}_z)_Q)/2$. Thus, the model equations do not describe the full exact evolution of $S_{x,Q}$, but, nevertheless, the Hamiltonian and the stochastic terms are correct. We need numerical solutions of the exact and the model equations in order to further compare them.

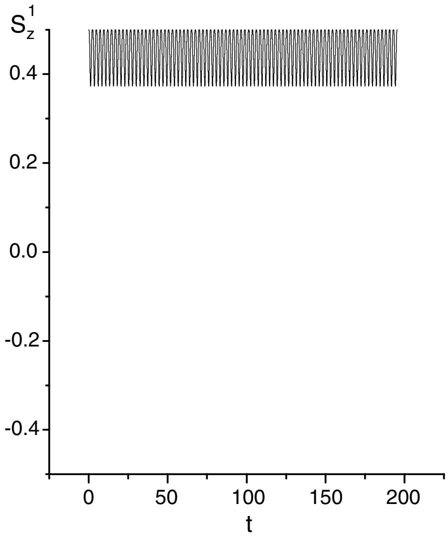
If the noise has components in the (x, y) plane and acts on both q-bits the model equations for $\mathrm{d}S^{1,2}_{x,y,z,Q}$ differ from the exact quantum equations. The relation between the two has to be studied by comparison of the numerical solutions.

As a final example, we consider the Hamiltonian used in [15] to model a pair of charge q-bits in an electromagnetic environment given by

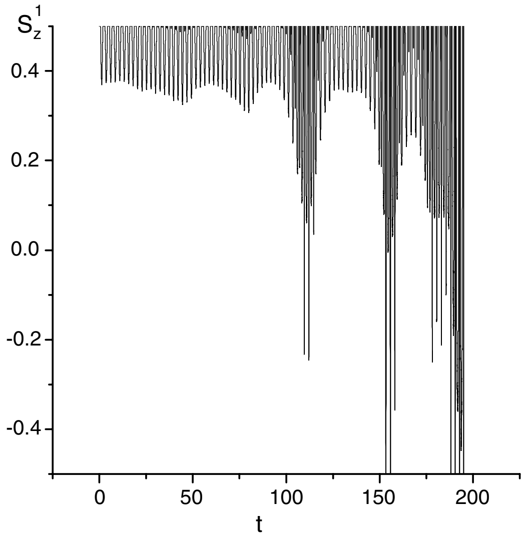
$$\hat{H} = -\mu\hat{S}^1_y\hat{S}^2_y - \hat{S}^1_x - \hat{S}^2_x + c\hat{S}^1_z + c\hat{S}^2_z + H_B, \tag{51}$$

where the first three terms correspond to the two q-bits, H_B is the Hamiltonian of the environment and the environment operators are again proportional to the $\hat{S}_z$ components.

Numerical solutions of the corresponding stochastic phase-space equation indicate conclusions which are qualitatively similar to those that have been obtained in [15] by approximate solutions of the Bloch–Redfield master equation for the reduced density matrix. For example, consider the dynamics of $\left(\hat{S}^1_z\right)_Q$ generated by the solution of the SDEs with the initial condition $(p_1, q_1, p_2, q_2) = (0, 0, 0, 0)$ corresponding to the state $|1/2, 1/2\rangle \bigotimes |1/2, 1/2\rangle$. If the noise is zero, the values of $\left(\hat{S}^1_z\right)_Q$ oscillate regularly with constant amplitude and in a domain, which includes the value at $|1/2, 1/2\rangle \bigotimes |1/2, 1/2\rangle$, i.e. 1/2 (see figure 1(*a*)). For a nonzero noise the behaviour of $\left(\hat{S}^1_z\right)_Q(t)$ is different (see figure 1(*b*)). It starts to oscillate with an increasing amplitude and after some time, proportional to the ratio between the spin coupling and noise, $\left(\hat{S}^1_z\right)_Q(t)$ oscillates irregularly through all possible values, i.e. from 1/2 to −1/2. This can be interpreted as an oscillatory dumping of the survival probability of the initial state $|1/2, 1/2\rangle \bigotimes |1/2, 1/2\rangle$. This is qualitatively consistent with the results presented in [15].



(*a*)



(*b*)

Figure 1. Illustration of the oscillations of the Q symbol $(\hat{S}_z^1)_Q(t)$ of z component of the first q-bit for the Hamiltonian (51) for (*a*) no noise $c = 0$ and (*b*) small noise $c = 0.029$. In both cases $\mu = 6$ and the initial condition in the phase space corresponds to the state $|1/2, 1/2\rangle \bigotimes |1/2, 1/2\rangle$.

7. Summary

In this paper, we introduced a phase-space model of a multi q-bit system in interaction with the environment. The original quantum system in a pure state is described by a linear stochastic Schrödinger equation. In order to obtain the phase-space model, we employed the path integral form of the coherent-state representation of the stochastic propagator. The stationary exponent approximation is used to obtain the complex Ito stochastic differential equations for the canonical variables of the phase space of the multi q-bit system. These equations are only an approximation, but in the case of the isolated system the phase-space Hamiltonian equations give Liouville equation for the Q symbols of the components $\hat{S}^i_{x,y,z}(t)$ which coincides with exact quantum equation, owing to the fact that the Hamiltonian is a multilinear expression of the generators $\hat{S}^i_{x,y,z}$. This motivated us to study the approximate model of the open system.

We have considered some simple and typical forms of the environment operators for which the approximate phase-space model can easily be compared with the exact quantum equations. We concluded that the model correctly simulates averaged evolution of $S^i_{x,y,z;Q}(t)$ of the ith q-bit if the q-bits other then the ith one are coupled to the environment, and the ith one is coupled only to the other q-bits but not to the environment. Furthermore, if the environment operators are given by the z components $\hat{S}^i_z$ then the model equations correctly simulate $S^i_{z;Q}(t)$. Evolution of the other components $S^i_{x,y;Q}(t)$ is not correctly described by the model because the drift terms in the approximate and the exact quantum equations are different, although the Hamiltonian and the stochastic terms are equal. If the environment operators have components in the (x, y) plane, then the model and the exact equation for $S^i_{x,y,z;Q}(t)$ differ in both the drift and the diffusion terms. Then, the relation between the phase-space model and the exact quantum system needs to be studied by comparison of the numerical solutions.

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